Equations of Motion in a Quantum-mechanical Theory of Gravity

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Abstract

An earlier paper [1] presented a gravity theory based on the optics of de Broglie waves rather than curved space-time. While the universe's geometry is flat, it agrees with the standard tests of general relativity. A second paper [2] showed that, unlike general relativity, it agrees with Doppler tracking signals from the Pioneer 10 and 11 space probes. There a gravitational acceleration equation plays an important role, accounting for the relative motions of Earth and the probes. Here it's shown that equation also describes Mercury's orbit.

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1 Introduction

A previous paper [1] introduced a theory of gravity based on the optics of de Broglie waves, without curved space-time. While agreeing with the standard experimental tests of general relativity, the theory is inherently quantum-mechanical. A second paper [2] showed that, unlike general relativity, it agrees with the observed motions [3] of the Pioneer 10 and 11 space probes.

Many-body systems are described by a gravitational acceleration equation derived in the first paper. That was used to calculate a perturbation of the Moon's orbit observed in the lunar ranging experiment. And subsequently to obtain the Pioneer motions. It also gives Mercury's observed orbit. Since the acceleration equation is a pivotal one, we'll demonstrate that here.

Like the quantum mechanics ("wave mechanics") of de Broglie and Schrödinger, the development of this gravity theory was guided by the optical-mechanical analogy [4]. In optics, to fully describe the propagation of light waves and photons, it's necessary to calculate wave amplitudes, as given by Huygens' principle. In the short-wavelength limit, where diffraction and interference effects can be ignored, geometrical optics can be used.

De Broglie [5] and Schrödinger [6] found the same holds for matter and matter waves. Where the de Broglie wavelengths of particles are sufficiently short, quantum mechanics reduces to ordinary Hamiltonian mechanics. As Hamilton discovered earlier, the latter is directly analogous to geometrical optics.

Rather than curving space-time, the effect of gravitational potentials in this theory is a slowing of quantum mechanical waves – both matter and light. (Where Einstein assumed a constant speed of light and variable space and time, the assumption here is the opposite.) From de Broglie, the velocity V of matter waves is

$$V = \frac{c^2}{v} \tag{1}$$

where c is the speed of light, and v is the velocity of an associated particle or body.

Here the speeds of light and de Broglie waves in a gravitational potential Φ_g are given by

$$\frac{c}{c_0} = \frac{V}{V_0} = e^{2\Phi_g/c_0^2} \tag{2}$$

where the subscript 0 represents the same quantity in the absence a gravitational potential. The reduced wave velocities are manifested equally in their wavelengths and frequencies, as

$$\frac{\nu}{\nu_0} = \frac{\lambda}{\lambda_0} = e^{\Phi_g/c_0^2} \tag{3}$$

Since the frequencies of de Broglie waves in atoms determine the rates of clocks, and their wavelengths the sizes measuring rods, both diminish in gravitational potentials. Consequently, in accord with Poincaré's principle of relativity, the reduced wave velocities aren't apparent to a local observer. Time and space themselves are unaffected, and the latter remains isotropic, in agreement with the observed large-scale flatness of the universe.

It's pointed out, for example by Wheeler [7], that not only elementary particles, but atoms and macroscopic objects have de Broglie wavelengths. An atomic electron is subject to the condition that its orbit is an integral number of those wavelengths. In this theory, the same condition applies in principle to orbits of astronomical bodies.

Of course the h/p de Broglie wavelength for an orbiting body is many orders of magnitude shorter than an electron's. And since an orbit's size is also much larger, its quantization is unobservable. It follows the short-wavelength approximation can be used to describe astronomical orbits, by the methods of geometrical optics. (Determining trajectories orthogonal to the de Broglie wavefronts.) That was found to give an orbit equation which agrees with Mercury's [1].

The relativistic Lagrangian for a charged particle in electromagnetic potentials is

$$L = -m_0 c^2 \sqrt{1 - v^2/c^2} - q(\Phi - \mathbf{v} \cdot \mathbf{A}/c) \tag{4}$$

where m_0 is its rest mass, q the charge, and Φ and A are the scalar and vector potentials. This is related directly to its de Broglie frequency [1]. In this theory, the Lagrangian for a particle or body in a gravitational potential becomes

$$L = \left(-m_{00} c_0^2 \sqrt{1 - v^2/c^2} - q_0(\Phi_0 - \mathbf{v} \cdot \mathbf{A}_0/c)\right) e^{\Phi_g/c_0^2} \tag{5}$$

with m_{00}, q_0, Φ_0 and A_0 the corresponding quantities in the absence of a gravitational potential.

The resulting Euler-Lagrange equation of motion for a body in a central gravitational field gives the same first-order differential equation of the orbit found by de Broglie wave optics [1]. From that, this second-order equation was derived

$$\frac{d^2 r}{d\theta^2} = \frac{2 - \mu/r}{r} \left(\frac{dr}{d\theta}\right)^2 + r - \mu - \frac{r^2 \mu\, e^{4\mu/r}}{k^2} \tag{6}$$

where r and θ are polar coordinates. Here μ is given by

$$\mu = GM/c_0^2 \tag{7}$$

with G the gravitational constant and M the mass of the central body. And k is a constant of the orbit

$$k = r^2 \dot{\theta}\, e^{4\mu/r}/c_0 \tag{8}$$

with the dot indicating time differentiation.

For comparison, in this notation the corresponding Newtonian equation is

$$\frac{d^2 r}{d\theta^2} = \frac{2}{r} \left(\frac{dr}{d\theta}\right)^2 + r - \frac{r^2 \mu}{k^2} \tag{9}$$

where k again is a constant of the orbit

$$k = r^2 \dot{\theta}/c_0 \tag{10}$$

representing the body's conserved angular momentum divided by c_0.

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The Lagrangian of Eq. (4) gives the Lorentz equation for the electromagnetic force $\mathbf{F}$ on a charged particle

$$\mathbf{F} \;=\; q\left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\right) \tag{11}$$

where the electromagnetic potentials are translated into the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. This equation can be used to determine the motions of arbitrary systems of charges, when their energies aren't conserved. No comparable equation exists for gravity in general relativity. Instead, an approximate many-body Lagrangian is used to describe many-body systems [8].

In this non-metric theory, the Lagrangian in Eq. (5) *does* give a gravitational counterpart of the Lorentz force equation [1]

$$\mathbf{a} \;=\; -\nabla\Phi_g\left(e^{4\Phi_g/c_0^2} + \frac{v^2}{c_0^2}\right) + \frac{4\mathbf{v}}{c_0^2}\left(\frac{d\Phi_g}{dt}\right) \tag{12}$$

where $\mathbf{a}$ represents a body's acceleration. (Multiplying both sides by its mass would give a gravitational force.) Like Eq. (11), this can be used iteratively to calculate the motions of arbitrary systems of bodies whose energies aren't conserved. Below we'll derive an orbit equation from this equation of motion, and compare it to that already obtained from de Broglie wave optics and the Euler-Lagrange equations of motion.

2   Orbits from the Acceleration Equation

For a body orbiting in a central gravitational field, from Eq. (12), its radial acceleration can be expressed in isotropic polar coordinates as

$$\ddot{r} - r\dot{\theta}^2 \;=\; -\nabla\Phi_g\left(e^{4\Phi_g/c_0^2} + \frac{r^2\dot{\theta}^2 + \dot{r}^2}{c_0^2}\right) + \frac{4\dot{r}}{c_0^2}\left(\dot{r}\,\nabla\Phi_g\right) \tag{13}$$

(See Becker [9].)

In this theory, the gravitational potential due to a stationary spherical body is just

$$\Phi_g \;=\; -GM/r \tag{14}$$

From the notation of Eq. (7),

$$\Phi_g/c_0^2 \;=\; -\mu/r \tag{15}$$

and the gradient of the gravitational potential is

$$\nabla\Phi_g \;=\; c_0^2\,\mu/r^2 \tag{16}$$

After these substitutions, Eq. (13) becomes

$$\ddot{r} - r\dot{\theta}^2 \;=\; -\frac{c_0^2\mu}{r^2}\,e^{-4\mu/r} - \mu\dot{\theta}^2 + \frac{3\mu\dot{r}^2}{r^2} \tag{17}$$

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The orbiting body's radial velocity can be expressed in terms of $\dot{\theta}$ as

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} \tag{18}$$

Taking the time derivative,

$$\ddot{r} = \frac{d}{dt}\left(\frac{dr}{d\theta}\dot{\theta}\right)$$

$$= \frac{d^2r}{d\theta^2}\dot{\theta}^2 + \frac{dr}{d\theta}\frac{d\dot{\theta}}{dr}\dot{r} \tag{19}$$

Again, this theory gives a conserved quantity k for an orbiting body, similar to angular momentum. Rearranging Eq. (8) for k gives

$$\dot{\theta} = \frac{k\,c_0\,e^{-4\mu/r}}{r^2} \tag{20}$$

Differentiating that with respect to r, then putting the result in terms of $\dot{\theta}$,

$$\frac{d\dot{\theta}}{dr} = -\left(\frac{2}{r} - \frac{4\mu}{r^2}\right)\dot{\theta} \tag{21}$$

Substituting for $d\dot{\theta}/dr$ and $\dot{r}$ in Eq. (19), we get

$$\ddot{r} = \left(\frac{d^2r}{d\theta^2} - \left(\frac{2}{r} - \frac{4\mu}{r^2}\right)\left(\frac{dr}{d\theta}\right)^2\right)\dot{\theta}^2 \tag{22}$$

After substituting for $\dot{r}^2$ in Eq. (17), we also get

$$\ddot{r} = -\frac{c_0^2\mu e^{-4\mu/r}}{r^2} + r\dot{\theta}^2 - \mu\dot{\theta}^2 + \frac{3\mu}{r^2}\left(\frac{dr}{d\theta}\right)^2\dot{\theta}^2 \tag{23}$$

Equating the right sides of the last two equations, dividing by $\dot{\theta}^2$, and rearranging gives

$$\frac{d^2r}{d\theta^2} = \frac{2 - \mu/r}{r}\left(\frac{d^2r}{d\theta}\right)^2 + r - \mu - \frac{c_0^2\mu e^{-4\mu/r}}{r^2\,\dot{\theta}^2} \tag{24}$$

Finally, substituting for $\dot{\theta}^2$ from Eq. (20) gives the second-order differential equation of the orbit found previously, Eq. (6).

The initial paper showed Eq. (6) is the derivative of this one

$$\frac{dr}{d\theta} = \frac{r\sqrt{r^2\left(e^{4\mu/r} - \frac{E_{00}^2}{E^2}e^{2\mu/r}\right) - k^2}}{k} \tag{25}$$

where E and E_{00} are constants of the orbit, corresponding to the body's total relativistic energy and its rest energy in the absence of a gravitational potential. This first-order equation was derived from both de Broglie wave optics and the Euler-Lagrange equations of motion. And it was shown to agree with Mercury's orbit [1].

3 Conclusions

Mercury's orbit remains one of the strongest tests of general relativity. The same orbital precession is predicted by this quantum-mechanical theory. We've arrived at a single orbit equation by three routes now: Directly from de Broglie wave optics, the Euler-Lagrange equations of motion, and Eq. (12) for gravitational acceleration.

Unlike the Euler-Lagrange equations, the acceleration equation describes the motion of bodies whose energies aren't conserved. It also describes the relative motions of Earth and the Pioneer 10 and 11 space probes [2], which are unexplained by general relativity.

Acknowledgments

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